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The Parametric Approach to the Architectural Problem of Projecting Higher Dimensional Hypercubes to R^3, and its Correlation with the Tessellation of a Plane

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Abstract

This paper presents a method for approaching three dimensional models for n dimensional hypercubes through polar zonohedra, which – under certain conditions – constitute orthographic isometric projections of such hypercubes. The paper then goes on to present certain sections of the solids in question and the creation of tessellations on the plane. In order to design the zonohedra, use was made of the Rhino program, which combined with the Grasshopper routine allows for the parametric control of the geometric structure of the solid. In other words, it shows – through the proper manipulation of the design algorithm – how zonohedra are produced, constituting projections of higher dimensional hypercube spaces in three dimensional space. Subsequently, the sections of the zonohedra create planar tessellations on the planes, which change depending on how the *n* degree of the zonohedron changes. This results in a table that juxtaposes projections of hypercubes in three dimensional space and tessellations of a plane, some of which are already known, thus suggesting some sort of correlation between them. This study serves as a formulation of the architectural question surrounding the concept of the projection of polyhedra in general dimension on a plane and suggests an approach involving the parametric control of structures, thus bypassing – to a certain degree – the need for supervision. It also provides an answer to the general question regarding the contemporary role of geometry in the education of architects, which focuses mainly on the gradual detachment of the architect from the need to constantly monitor the produced form. The study presented in this paper is based on the post-doc research made by Nikos Kourniatis under the research funding program Thalis. Ioannis Emiris was the supervisor professor.

From platonic solids to polar zonohedra

The Minkowski sum of n vectors in space is a convex polyhedron [3] with n(n 1) faces, where n is the number of the different directions of the vectors[4]. If the vectors are equal in size, then the faces of the convex polyhedron will be shaped as rhombi and the polyhedron will constitute an equilateral zonohedron.

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[3] Corrado De Concini, Claudio Procesi, Topics in Hyperplane Arrangements, Polytopes and Box-splines, Springer, 2010, p.45
[4] W.W. Rouse Ball, H. S. M. Coxeter, Mathematical Recreations and Essays, New York: The Macmillan Company, pp. 141-144, 1947

Equilateral zonohedra are considered as 3 dimensional projections of n dimensional hypercubes[5]. The more symmetrised the initial vectors are, the more symmetrical the resulting zonohedra will be.

Fig. 1

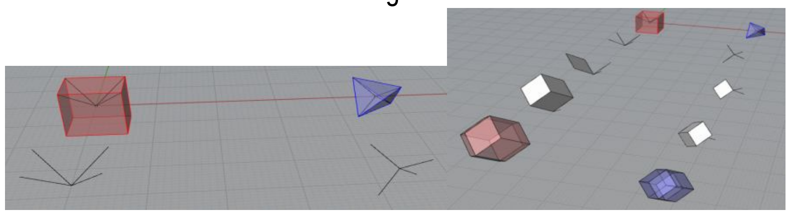


Fig. 2

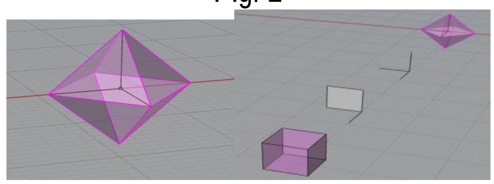


Fig. 3

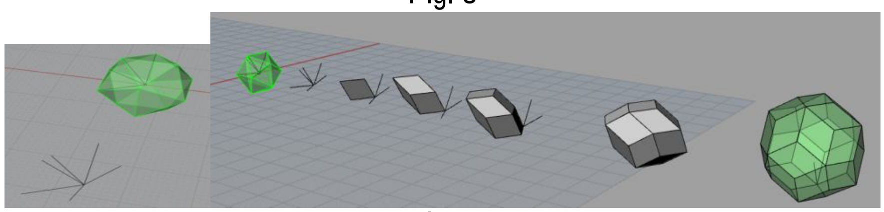
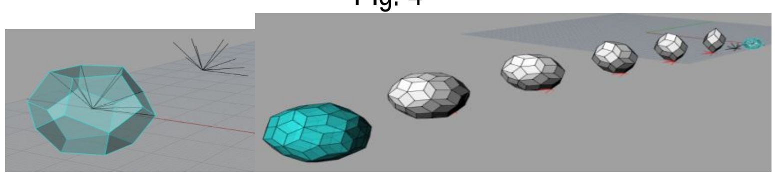


Fig. 4



The most symmetric zonohedra are those resulting from platonic solids[6], with the vectors directed towards the linear segments, which project the vertices of each polyhedron from its centre. Thus, the cube and regular tetrahedron result in a rhombic dodecahedron (Fig. 1), the regular octahedron results in a cube (Fig. 2), the regular icosahedrons results in Kepler's golden rhombic triacontahedron (Fig. 3), whose faces are rhombi with a diagonal ratio equal to the golden ratio Φ, and the regular dodecahedron results in rhombic enneacontahedron (Fig. 4), whose faces consist of two types of rhombi (60 of the one type and 30 of the other). The above zonohedra constitute 3D models of hypercubes.[7]

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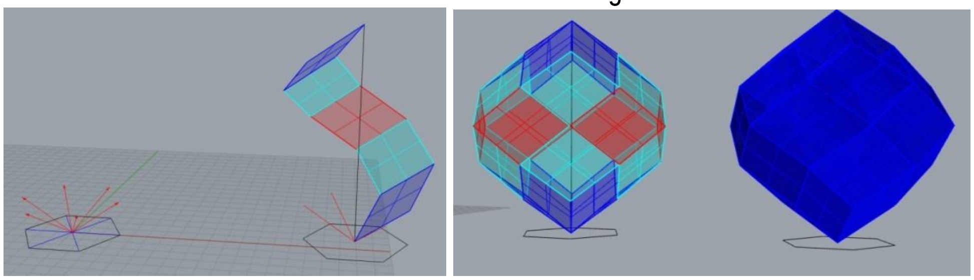
5 CRC Concise Encyclopedia of Mathematics, p. 3240
W.W. Rouse Ball, H. S. M. Coxeter, Mathematical Recreations and Essays, New York: The Macmillan Company, p. 140, 1947

6 W.W. Rouse Ball, H. S. M. Coxeter, *Mathematical Recreations and Essays,* New York: The Macmillan Company, New York: The Macmillan Company, p. 142, 1947
Russell Towle, "Polar Zonohedra", The Mathematica Journal, 1996, p. 8

7 Russell Towle, "Polar Zonohedra", The Mathematica Journal, 1996, p.9

Fig. 5



Polar zonohedra form a unique category of zonohedra. Let us take a regular n gon in the plane and line segments ε_1, ε_2,..ε_ν, which connect the centre O of the polygon to its vertices. Then let us take the equal vectors δ_1, δ_2,…δ_ν, with O being the common starting point, which are projected in the plane of the n gon by ε_1, ε_2,…ε_ν. The zonohedron resulting from the Minkowski sum of vectors δ_1, δ_2,…δ_ν, which is called a polar zonohedron, is a convex polyhedron whose n fold axis is the vertical line in the centre of the plane of the polygon with n(n 1) rhomboid faces laid out in zones around the axis (Fig. 5). The algorithm in figure 6 allows us to create polar zonohedra by controlling the number of vectors, their inclination to the plane of each polygon and their size. This results in a variety of forms (Fig. 7). When several of the sides of a regular polygon are infinite, then the zonohedron leans towards a surface of revolution. It has been observed that [8] when the angle of inclination of the vectors to the plane of the polygon is 36.264°, the n polar zonohedron is considered a 3D orthographic isometric projection of the n hypercube.

Fig. 6

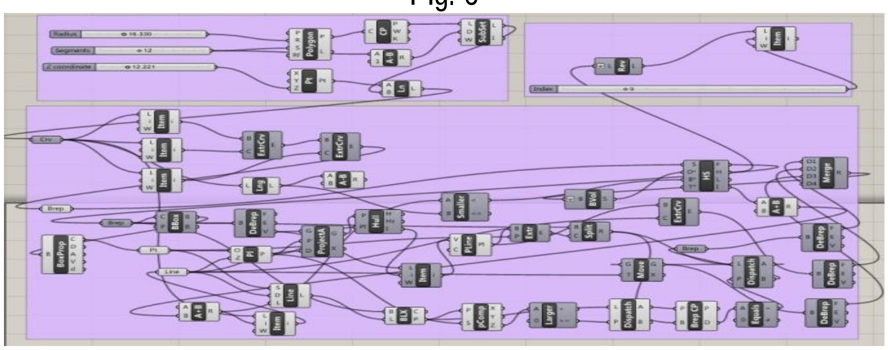
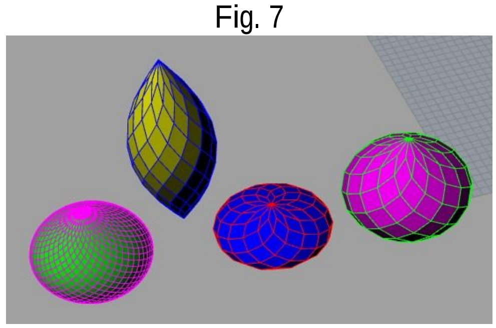


Fig. 7



[8] Russell Towle, "Polar Zonohedra", The Mathematica Journal, 1996, p.9

The description of the algorithm

We will now provide a brief description of the function of the design algorithm, which has been designed to start off from a regular n gon and to result in zonohedra, which constitute Euclidean models of higher dimensional cubes, allowing for control of the inclination angle of the vectors to the horizontal plane, and also of the number of different directions that form the zonohedron each time. The algorithm in question was essentially used to plan the described procedure, which refers more to the geometric structure of the produced object, and not so much to its form. This approach to the designed object is detached from the architect’s conventional need to control form.

1. Enter the initial data (polygon, edge size and height) into the design algorithm (Fig. 8) and control the desirable inclination angle of the vectors. Do not set the angle at 36.264° degrees from the start, which will result in orthographic isometric projections of hypercubes, so as to avoid limiting the variety of forms.
2. Thereafter, by selecting the first three vectors, the initial rhombohedron (3D model of the 3 cube) will be constructed using the Minkowski sum. This rhombohedron will be reset by the algorithm whenever the initial data is modified.

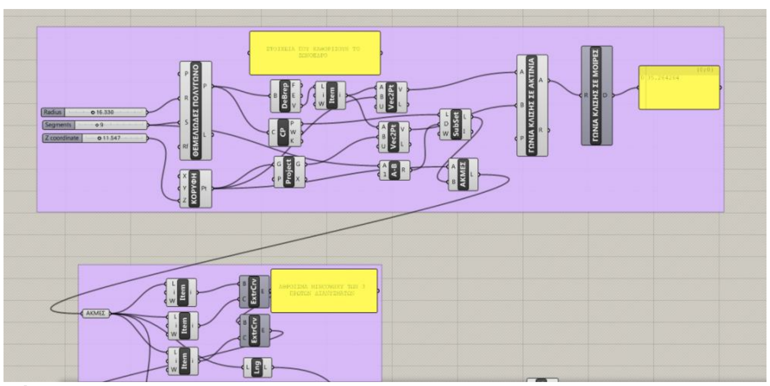


Fig. 8

3. The zonohedron is constructed on the basis of the following rationale: When a vector is added, a series of new faces is added in the direction of this vector. The algorithm initially creates a section in the centre of the polyhedron, perpendicular to the new vector. It then provides a parallel projection of the apparent outline of the polyhedron, which occurs if the polyhedron is projected parallel to the new vector. This results in a spatial polygonal line, beyond which the new faces are placed each time (Fig. 9).

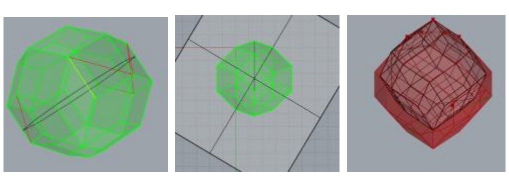


Fig. 9

4. Lastly, it divides the polyhedron into two sections based on this polygonal line and, once it sets and places the new faces, it reconnects the sections (Fig. 10).

Fig. 10

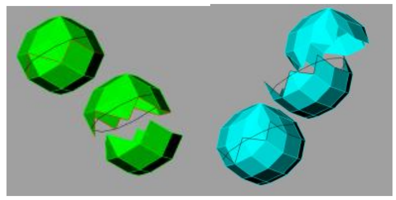
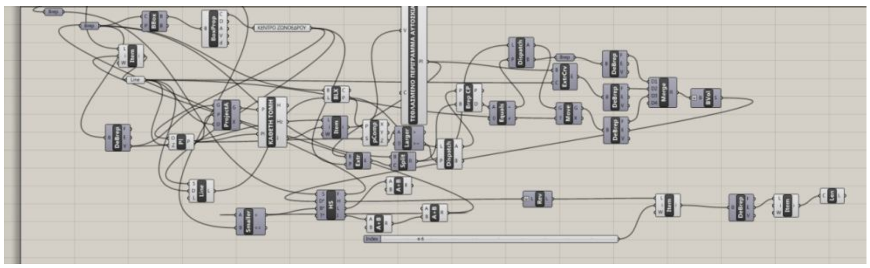


Fig. 11

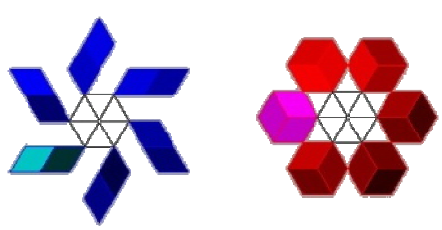


A second algorithm has also been designed, which controls the gaps and overlaps that may exist in the spatial tessellations of n cube models. This algorithm allows a plane, which can occupy all the positions in space (Fig. 11), to intersect the spatial tessellation and remove the section thereof that is above and below this plane, so that the structure of the tessellation can be visible from each position of the intersecting plane. With regard to these planar sections, a third algorithm determines their distances and number, so that we may control how close the resulting sections are and thus draw conclusions on the variations in the tessellation, depending on the movement of the intersecting plane.

The transition from smaller to larger dimensions

Considering that the cube can create a spatial tessellation[9], which – with the appropriate sections – can result in planar tessellations, we will explore the possibility of creating spatial structures and, by extension, planar tessellations from hypercubes by working with their three dimensional models.

Fig. 12



This exploration is also based on the fact that each n polar zonohedron and, consequently, each n hypercube can result from the composition of zonohedra[10] of a lower order.

[9] Gardner M., *The Sixth Book of Mathematical Games from Scientific American.* Chicago, IL: University of Chicago Press, 1984, pp. 183-184
[10] W.W. Rouse Ball, H. S. M. Coxeter, *Mathematical Recreations and Essays,* New York: The Macmillan Company, p. 147, 1947
David Wells, *The Penguin Dictionary of Curious and Interesting Geometry*, p. 275

Thus, for example, the rhombic triacontahedron (which is a 3D model of the 6 cube) can result as follows: Let us take the parallelohedra defined by two triads of the six vectors that determine the rhombic triacontahedron. The combination of two oblong and two oblate parallelohedra (Fig. 12) results in a rhombic dodecahedron (3D model of the 4 cube) (Fig. 13). This, along with the use of six additional parallelohedra, results in a rhombic icosahedron (3D model of the 5 cube) (Fig. 14). The rhombic icosahedron combined with 10 parallelohedra, results in a rhombic triacontahedron (which is a 3D model of the 6 cube) (Fig. 15).

Fig. 13

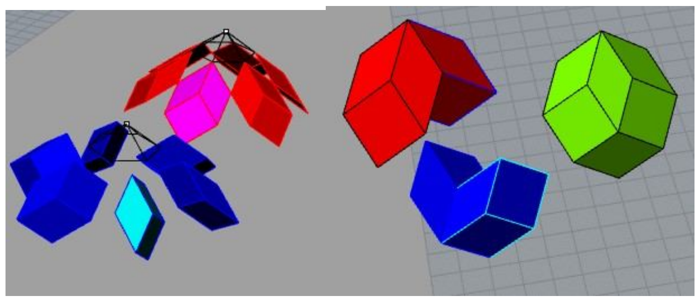
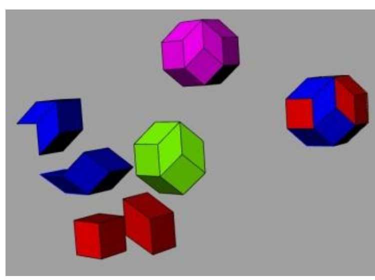
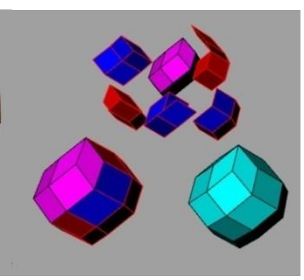


Fig. 14 Fig. 15

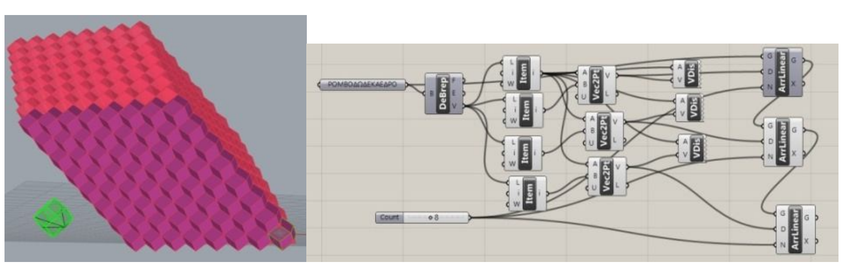




Spatial tessellation of 4 cube models

Out of the zonohedra, the rhombic dodecahedron (3D model of the 4 cube) can create a spatial tessellation[11], recurring in the direction of vectors δ1, δ2 and δ3 (Fig. 16).

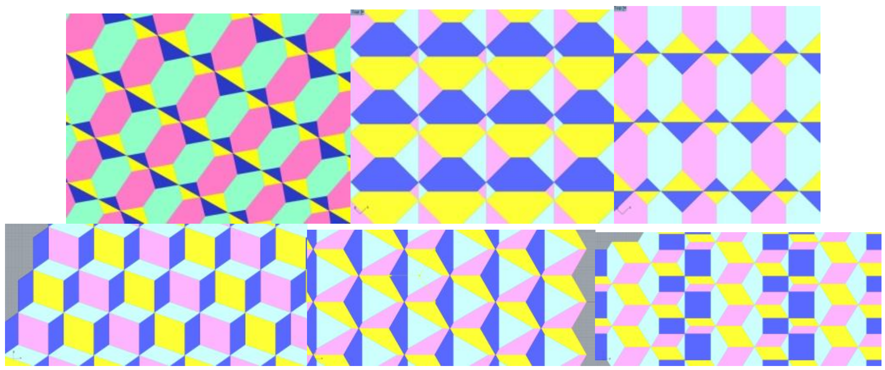
Fig. 16



[11] David Wells, *The Penguin Dictionary of Curious and Interesting Geometry*, p. 233

If we replace the rhombic dodecahedron with the 4 rhombohedra (3D models of the 3 cube) of which it is composed, we will get the same spatial structure, whose planar sections form periodic tessellations, as can be seen in figure 17.

Fig. 17



Spatial composition based on the 3D model of the 4 cube, while maintaining the rotational symmetry of the initial zonohedron

As we have seen, the rhombic dodecahedron (3D model of the 4 cube) creates a recurring spatial tessellation; however its structure lacks rotational symmetry. If we take the rhombic dodecahedron resulting from the Minkowski sum, four of the six vectors of which the rhombic triacontahedron (3D model of the 6 cube) is composed, combined with the corresponding rhombohedra, will result in a spatial structure that maintains the six fold rotational symmetry of the rhombic triacontahedron and can create a spatial tessellation.

Fig. 18

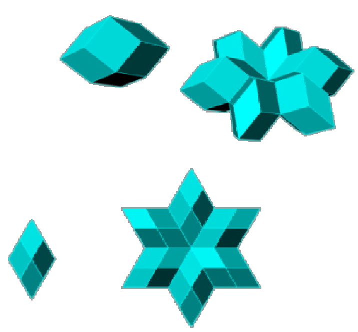


Fig. 19

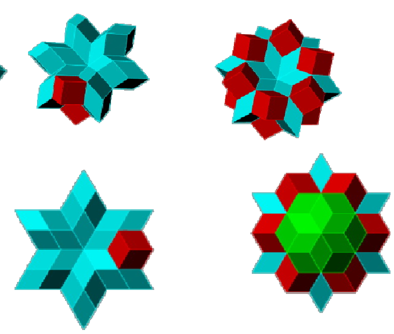


Fig. 20

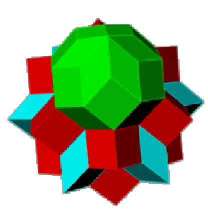
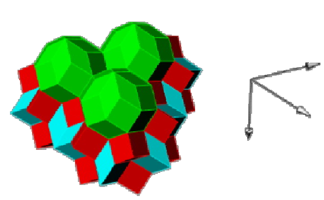
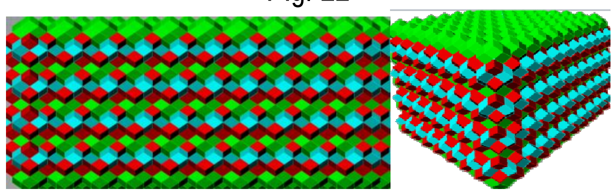


Fig. 21



In particular, the rhombic dodecahedron with a six fold rotational repetition leads to the creation of a spatial pattern (Fig. 18), which – if supplemented by six rhombohedra symmetrically arranged in relation to the same centre – forms a slot (Fig. 19) that can be filled precisely by a rhombic triacontahedron (Fig. 20). Through translations by vectors $\varepsilon 1$, $\varepsilon 2$ and $\varepsilon 3$, or through the appropriate rotations (Fig. 21), this spatial composition results in the spatial tessellation of the 3D models of the 4 cube and 6 cube (Fig. 22).

Fig. 22



The planar sections of this spatial tessellation lead to tessellations that transform from regular to semi regular and eventually to interesting periodic tessellations. Figure 23 presents successive parallel sections of this spatial structure and the resulting tessellations, with a plane running parallel to the xy plane.

Fig. 23

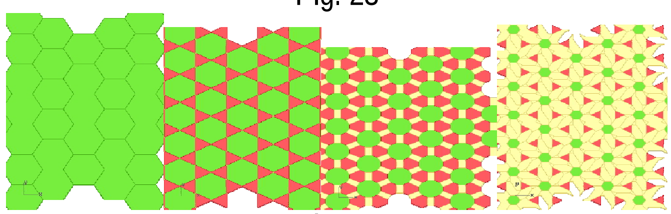
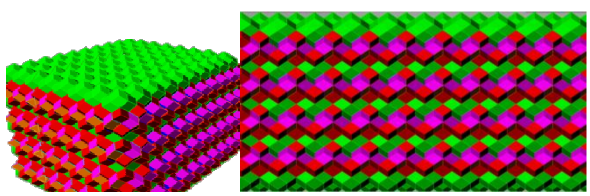
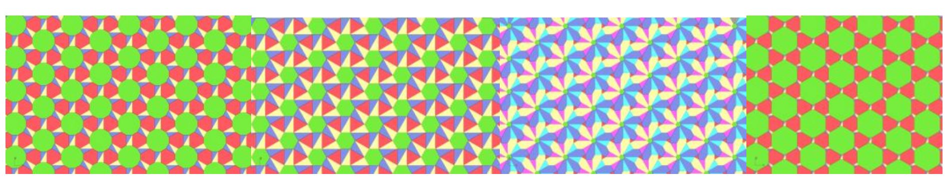


Fig. 24



The resulting patterns are even more interesting if the rhombic dodecahedra are replaced by the corresponding rhombohedra in various configurations (Fig. 24). Figure 25 presents planar tessellations resulting from the intersection of the spatial tessellation, parallel to the xy plane, while in figure 26 the sections are parallel to the xz plane.

Fig. 25



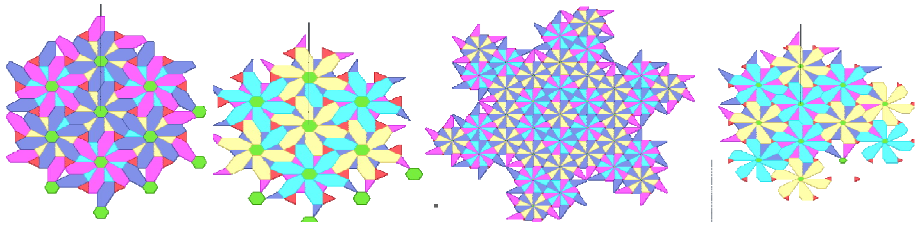
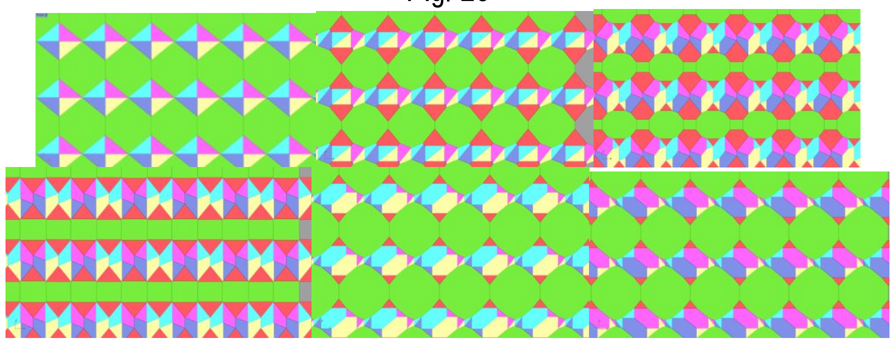


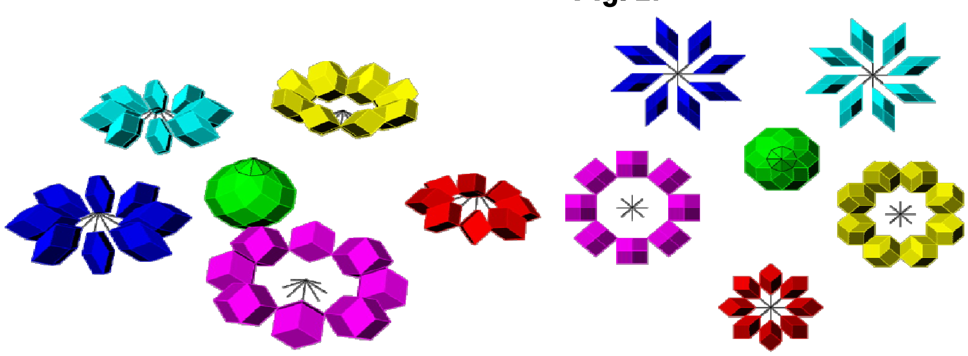
Fig. 26



Spatial composition based on the 3D model of the 8 cube, while maintaining the rotational symmetry of the initial zonohedron

Following the same rationale as that in the previous cases, we apply the algorithm to first construct the polar zonohedron corresponding to the 8 cube, starting from a regular octagon and the resulting vectors, so that the zonohedron constitutes an orthographic isometric projection of the 8 cube. Subsequently, by combining the vectors in fours, we construct the rhombic dodecahedra that we will use to enclose the 8 cube. The rhombohedra will be constructed in the same way (Fig. 27).

Fig. 27



The construction of the basic pattern, which will – through parallel translations – result in the spatial tessellation, is presented in figure 31. We initially enclose the 8 cube, whose horizontal projection is a regular octagon, with eight rhombic dodecahedra (3D models of the 4 cube) having a square plan view, by creating 8 polar arrays (Fig. 28). The gaps that are formed can be filled – with the same central symmetry – by another 8 rhombic dodecahedra (in yellow) (Fig. 29) and, subsequently, by 8 rhombohedra (3D models of the 3 cube, in red) (Fig. 30). This configuration is repeated symmetrically in relation to the horizontal plane of symmetry of the zonohedron. Finally, groups of 8 rhombic dodecahedra (in blue) complete the pattern and create slots for the repetition of the construction (Fig. 43).

Fig. 28 Fig. 29 Fig. 30

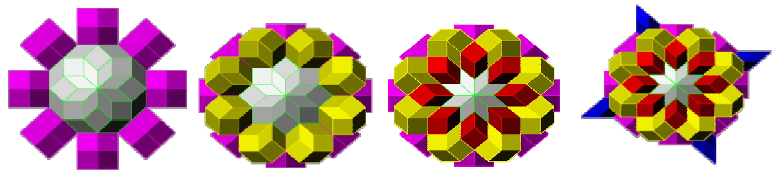
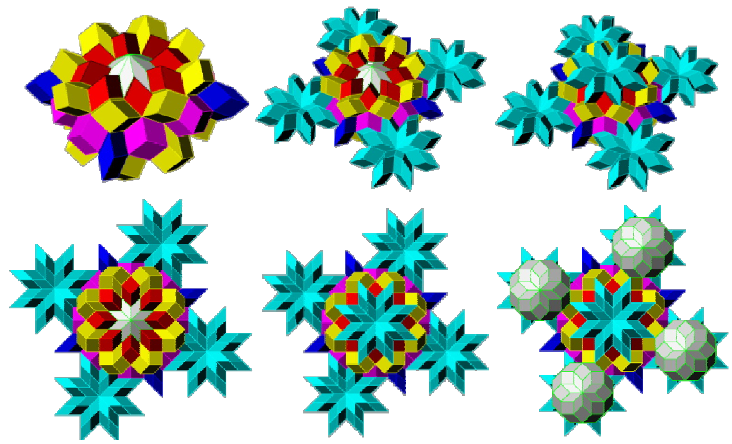


Fig. 31



The spatial tessellation resulting from the translation of the pattern has the form presented in figure 32, in layers. Figure 33 presents planar sections of this spatial structure, parallel to the xy plane.

Fig. 32

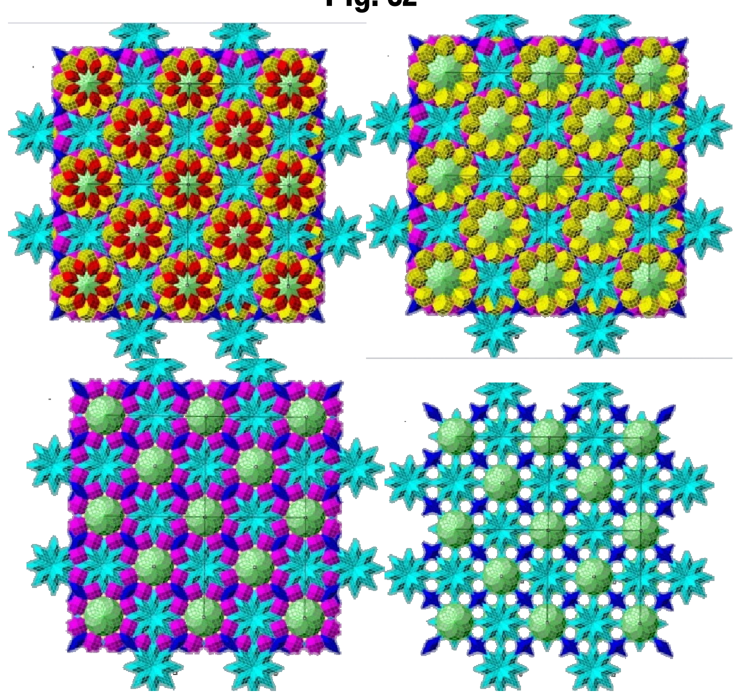
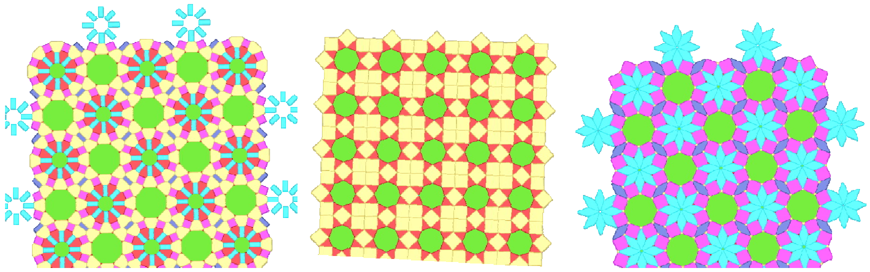


Fig. 33

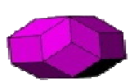


Conclusion

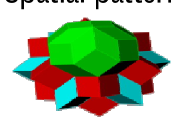
The process of constructing zonohedra and the tessellations that were presented lead to the conclusion that the polar zonohedra, which constitute Euclidean models of hypercubes in three dimensional space, can fill space and subsequently create planar tessellations, depending on the composition of the initial spatial pattern, which is repeated, thus filling the space. These tessellations can alternate from regular to semi regular and to periodic, depending on the position of the intersecting plane. The method followed shows architects an approach to the geometric structure of tessellation, which intertwines regular n gons with projections of solids in order to achieve the tessellation of a plane, in a way that they have never been associated before now. Table 1 below collectively presents the correlation between the zonohedron, the spatial pattern filling the space and the planar tessellations resulting from every such structure.

Table 1

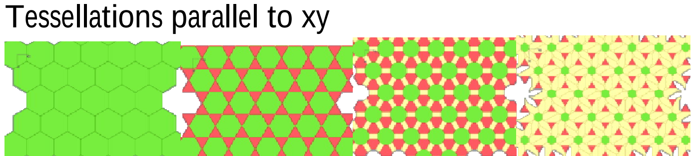
Model of the 4 cube

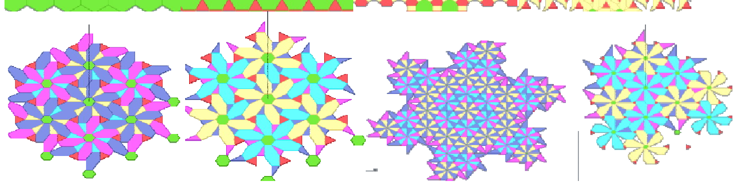


Spatial pattern

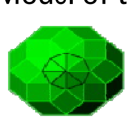


Tessellations parallel to xy

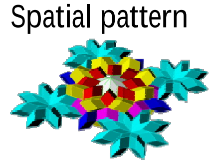




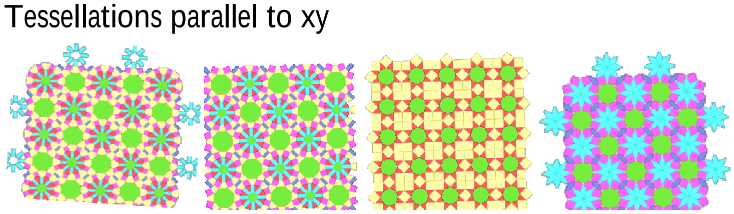
Model of the 8 cube



Spatial pattern



Tessellations parallel to xy



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