

'Structural Performance of Two-Hinged Islamic Arches'

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Abstract

This paper presents an investigation of Islamic arch forms and structural performance based on its geometrical formation. Arch form depends on several interrelated variables such as arc angles, the ratio of arc radius to arc span, and the height of the arch. These variables are controlled in Islamic architecture by the geometry. This paper provides insights for the geometric design of Islamic arches and outlines its structural analysis under different load cases. A parametric study of the buckling behavior of arches for several load cases, horizontal thrust, and bending moment will be analyzed. This paper aims to provide an understanding of the structural concepts of the arch form in Islamic architecture based on the significance of geometry in construction of Islamic arches. Arches described by geometry and proportions, play a crucial role in the composition and structure of the arch form. Typically design parameters are based on estimating force vectors to determine arch shape, assuming that they represent the architecture properly and are constant. To describe arch architecture in more detail and compare geometrical and non-geometrical states, geometric approach was validated. Arch forms are constructed from historical sections, based on geometric language, as well as the shape of the arch. The proposed approach may help to identify functional structural concepts within the arch, with homogeneous architectural geometric properties. The related results are discussed regarding the parameters essential for structural arch modeling and challenge aesthetic parameters of the arch-model.

Keywords: Islamic Arch, Arch Stability, Arch structure, Geometry

1. Introduction

The study of structural analysis and geometric principles of the design of traditional Islamic arch with regard to geometric proportion analysis, and the description of the syntheses of the arch structural concepts finds that it followed a pre-established geometric design with high accuracy and great precision. These findings support the argument that geometry was independently discovered and used in Islamic architecture and constitutes one of the most important design tools in architecture. Applications of this tool are presented with specific reference to examples in the field of modular arch structure. The case studies of a traditional arch prototype are used to explore the morphology of a traditional arch structural performance. This paper presents a structural investigation of Islamic arch geometry. Its structure and mechanics are significantly different from other arch forms because of its special geometrical form. Arch forms will be subjected to axial compression force, bending moment, torque, and shear stress. The investigation of the geometric models of arch structure is based on the geometric composition of the arch model evaluated by comparing different experimental tests to incorporate and determine the structural properties of the arch prototype.

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The aim of this study is to develop a technique that examines the structural analysis of proportional architecture of arches, and to determine how the arch architectural parameters change due to structural forces. To accomplish this, the arch types are investigated in compression and tension conditions. Arches were compared based on angles and curvatures. Additionally, the shape of the arch was analyzed because they are commonly used as a reference system. Local differences in arch angles, lengths, and curvatures, as well as changes in these parameters with different cases of loadings will be expected. The results are discussed regarding both their implications for established methods that estimate maximum force and the importance of their constructed architectural characteristics as input parameters in arch modeling.

2. Geometry of Islamic Arches

Geometry is an unseen presence in Islamic art and architecture. It is the visual representation of the mathematical patterns by which the human mind can comprehend order and harmony. This paper discusses proportions and geometry as a tool to achieve a performance oriented process in arch design, with specific focus on arch structural concepts. The key role played by geometry in arch is discussed in relation to proportions as a design process, in which evaluations based on structure and aesthetic criteria are integrated into the design process. This serves two objectives. Firstly, geometry in arch design was almost self-guiding methods of regular and aesthetically proved design. Secondly, Proportion geometry regulates primarily the *order* concepts, but at the same time it *mediates* between the two poles of order and diversity as well as between aesthetic levels. Arches described by geometry and proportions, plays a crucial role in the aesthetic and structure of the arch form. Usually, parameters based on estimate force vectors or lengths are used to determine arch shape, assuming that they represent the architecture properly and are constant. To describe arch architecture, geometric approach in the construction of arch forms from historical sections, based on geometric language that determine the shape of the arch is validated. The considered approach may help to identify functional homogeneous architectural properties. The results are discussed regarding the parameters essential for arch modeling and challenge structural force estimations that are based on the area or the arch-model.

Geometry thus determine the functional characteristics of arch and are therefore of interest when variations in arch architecture are studied. Arch architecture can differ greatly among arches within proportional systems. Centre angles may vary so greatly from arch to arch that even in rather simple arches they cannot be described by single parameters. Usually, arches are studied and investigated from the effects of the span division lines. The geometric techniques clearly have the advantage of depicting and thus allow looking at the arch structure which requires relatively proportional systems. Fig (1) represents the prototypes of Islamic arches drawn based on geometric division of the span line. Although the arch's curvature is correlated with the angle, the main parameters are presented through arch span and it is independent from the reference system in contrast to angles.

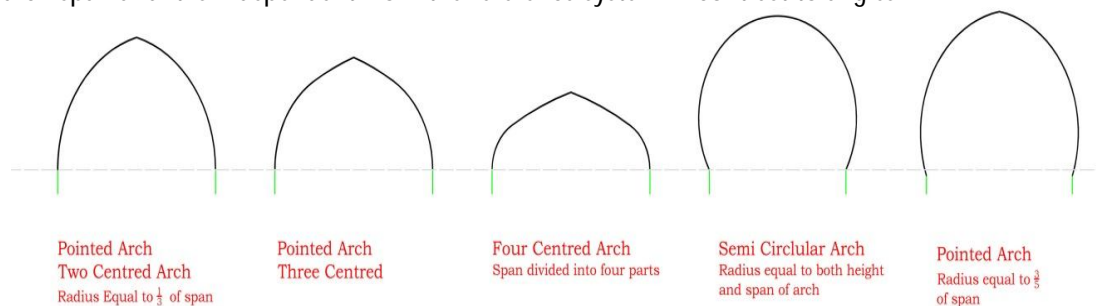


Fig. 1: Prototypes of Islamic Arches

The reconstructed arches were aligned at their center, the pointed arch, curvature coefficients were higher. Thus, arches were bent only slightly in the center but more curved towards the attachment sites. Great variation occurred within the arch due to the length portions the prototypes used in this study produce arches using the geometric modeling process. The layer-by-layer construction results in arch matrix that is homogeneous. Characterization of the matrix arch properties was necessary so that the model could accurately represent the structural properties.

Fig. 2 shows a matrix of Islamic arch. In arch design, it is recognized that the underlying arch-architecture plays an important role in structural mechanics of the arch. Arch models provide estimates of local tissue mechanics which can give insight into the mechanisms of arch structure as well as form a basis for inductive models of mechanically driven arch adaptation. The purpose of this study is to establish standard experimental validation method for assessing the accuracy of geometric arch models at both the apparent and structural levels based on arch-specific geometry.

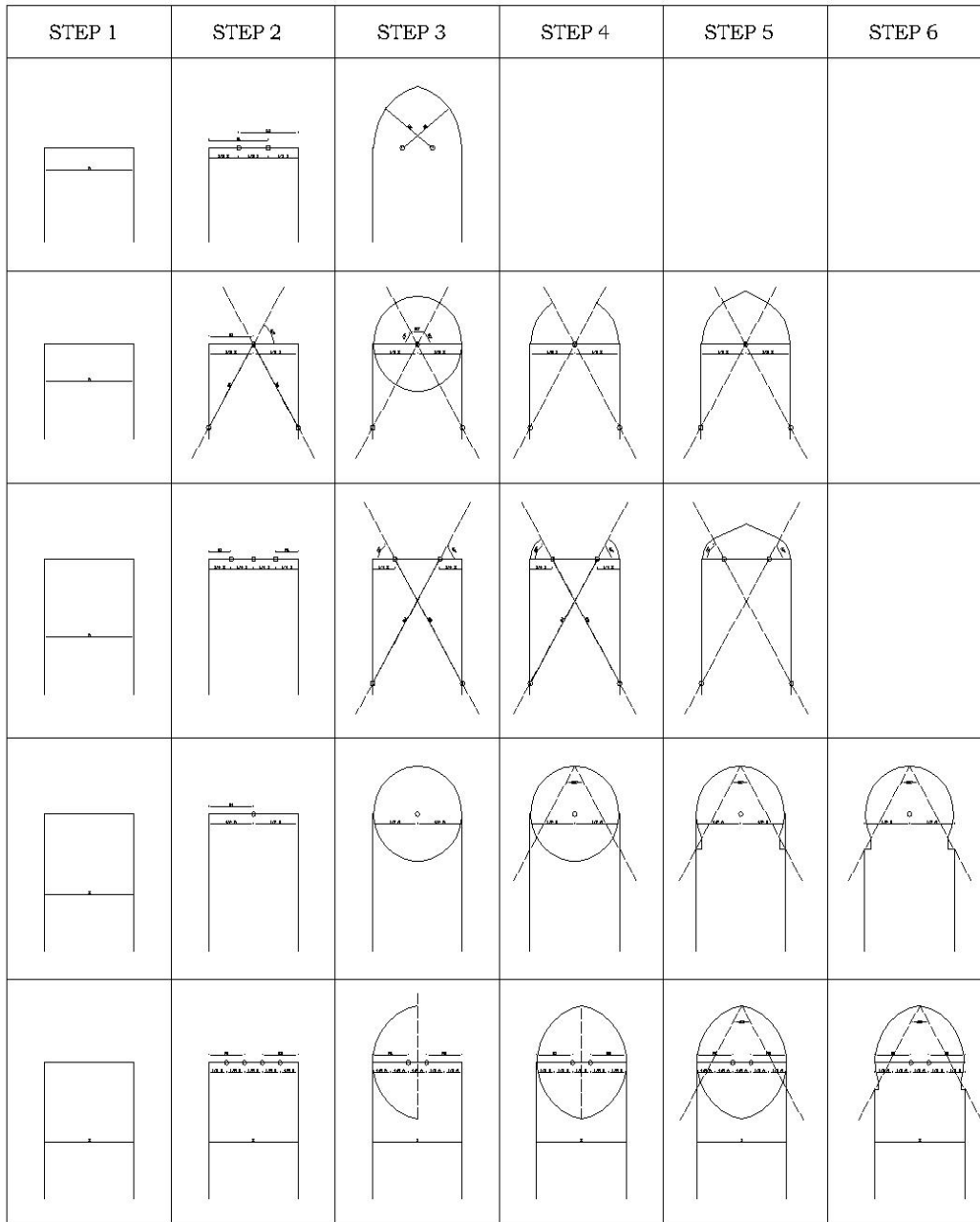


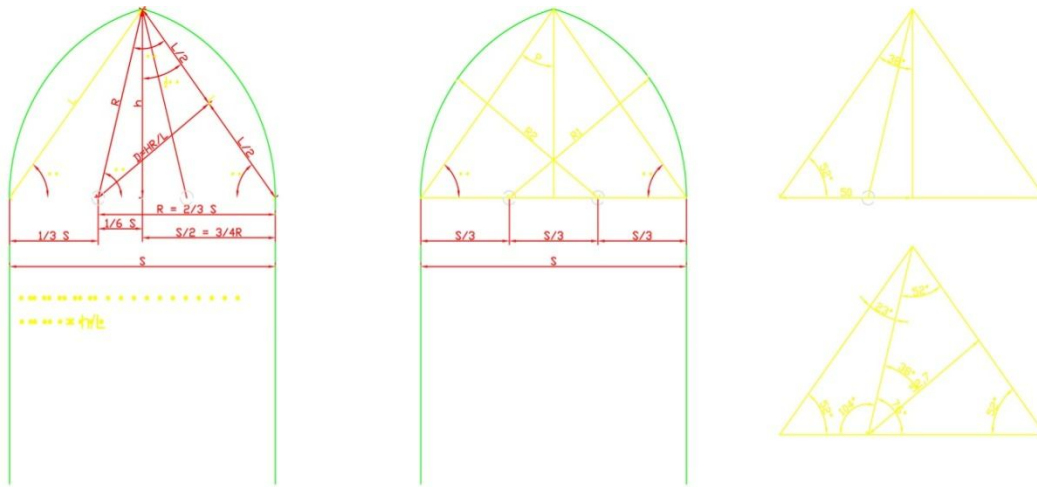
Fig. 2 Configuration Matrix of Islamic Arches

3. Structural Analysis

3.1 Geometry of Islamic Pointed Arches

For the purpose of this paper the two centered pointed arch (radius equal to 1/3 of span) is used as a basic case study. The pointed Islamic arch is composed of two circular arcs with independent radii.

Hence, there are many geometrical variables to be determined namely, the radius of the left arc which is $2/3$ of the span of the arch, $R1$, the radius of the right arc, and equal to $R2$, the angle (β) subtended by the two radiuses $R1$ and $R2$, is 76 degrees, the top angle of the arc, α , is 52 degrees, as well as the overall height of the arch, h . These variables are shown in (Fig 3). From the geometry of the arch the coordinates of the peak of the arch can be expressed as:



Pointed Arch
Two Centred Arch
Radius Equal to $\frac{1}{3}$ of span

Fig. 3 Geometric Configuration of Pointed Arches

$$\sin \beta / \sin \alpha = h / R \dots\dots\dots(1)$$

Equations (1) must be solved iteratively to determine all the geometric variables of a pointed semi arch, so as to simplify the design process. The graphs in Fig. 4 have been developed to determine the geometric variables for a specific height-to-half span ratio, R/H . The plots in Fig 4 shows the relationship between the ratios R/h for different values of angle α such that angle β can be determined from these independent variables.

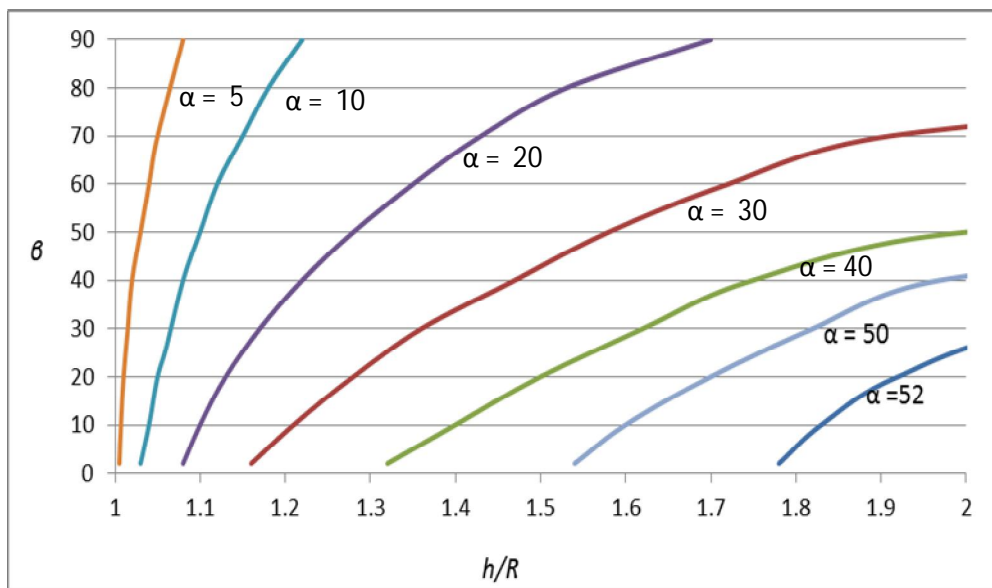


Fig 4: Relationship of H/R versus the angle β

Arches behave like two-dimensional beams spanning an open space, but unlike simple beams arches have a horizontal thrust resisting the tendency of the arch to open out. The common arch systems can be a three-hinged arch, a two hinged arch, or a fixed-fixed arch as shown in Fig. 5. For most common arch applications the two hinged arch is the most practical and is the arch system used in this study.

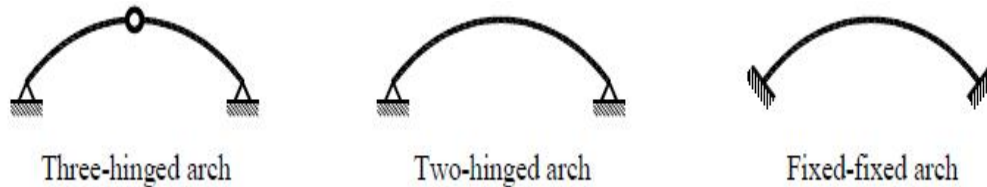


Fig. 5: Most Common Arch Types

3.2 Horizontal Thrust

The arch is assumed to be a two hinged arch with horizontal base reactions H , as shown in (Fig 6). To analyze this arch the principle of virtual work (Williams, 2009) or minimum strain energy (Timoshenko, 1930) is used giving the horizontal support reaction as:

$$H = \frac{\int Myds / EI}{\int y^2 ds / EI}$$

where M is the bending moment of the applied load case for a statically determinate simply supported arch, y is the distance from the base of the arch and represents the bending moment due to a unit horizontal load applied at the released support of the statically determinate simply supported arch, ds is the infinitesimal distance along the length of the arch, and EI is the bending rigidity of the arch (E being the modulus of elasticity and I the moment of inertia of the cross-section about the axis of bending). For the pointed Islamic arch, two circular coordinate systems are required as shown in Figure 3; the first for the lower part of the arch using θ as the variable and integrating the moments from $\theta = 0$ to $\theta = \pi/2$, and the second for the upper part of the arch using ϕ as the variable and integrating the moments from $\phi = a$ to $\phi = (a + \beta)$.

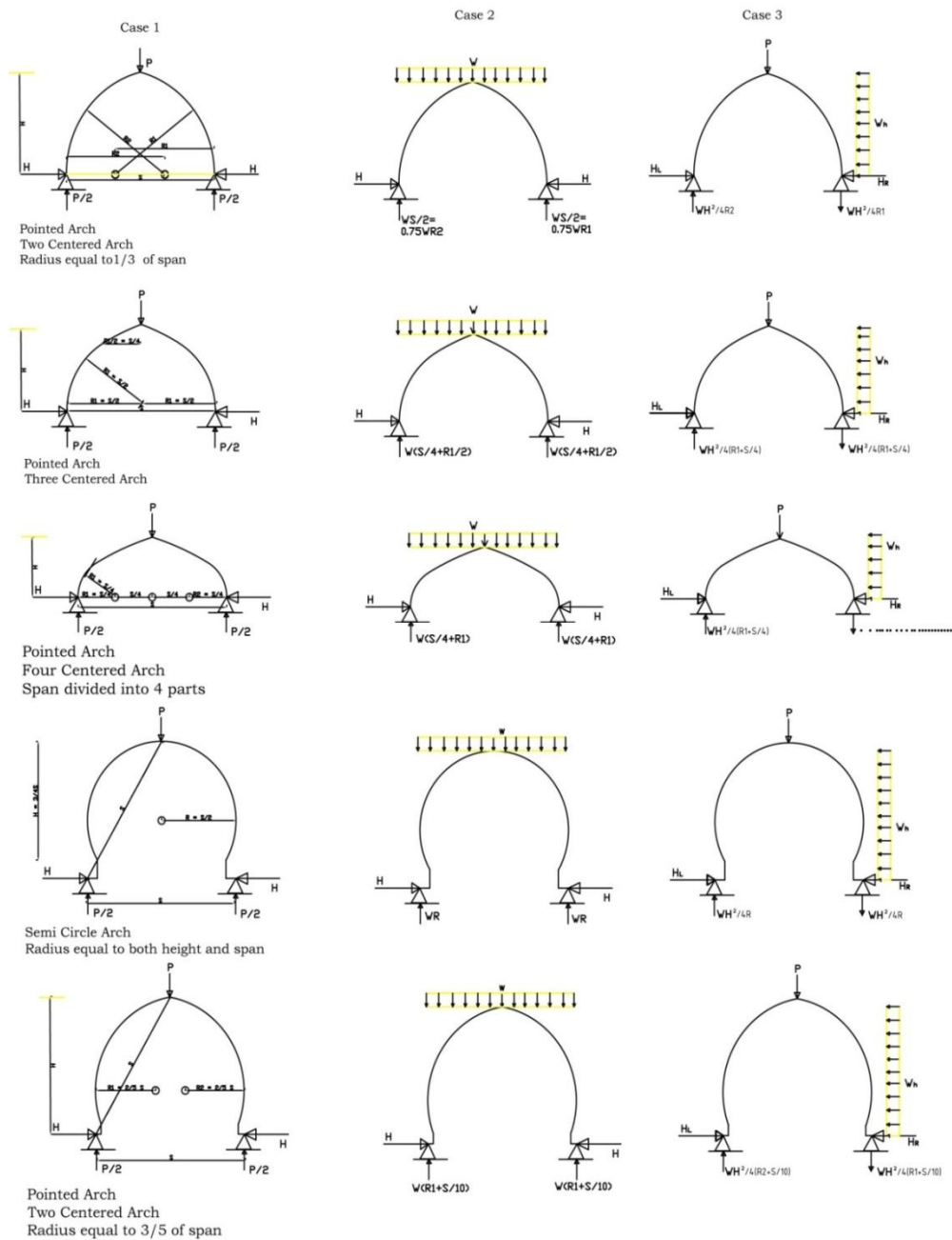


Fig 6: Load Cases Considered in Structural Analysis for Horizontal Thrust Reaction

The horizontal reaction that prevents the spread of the arch depends on the type of loading applied to the arch. Three cases of loading are analyzed namely a concentrated load P at the peak of the arch (mid span), a uniformly distributed vertical load acting along the horizontal projection w , and a uniformly distributed horizontal load acting along the vertical projection wh , which are shown in (Fig 6 , 7, 8 and 9).

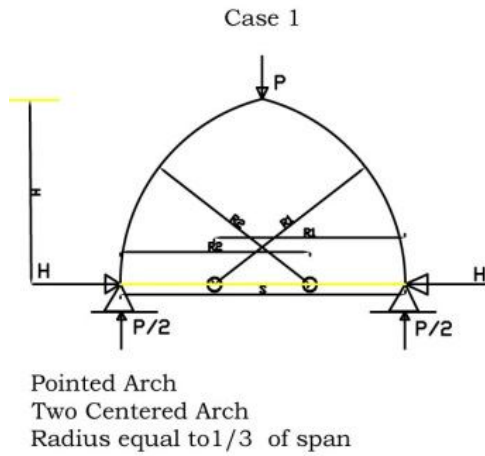


Fig 7: Horizontal Thrust for Case 1 Loading, Concentrated Load P at Mid span

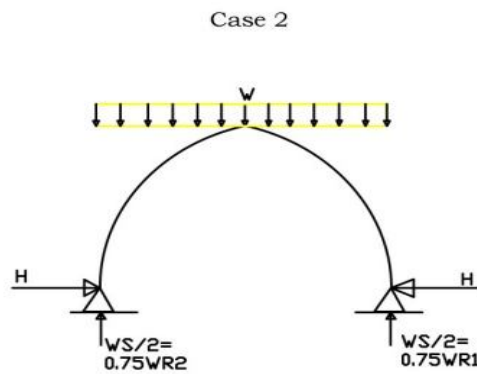


Fig 8: Horizontal Thrust for Case 2 Loading, Uniformly Distributed Vertical Load W

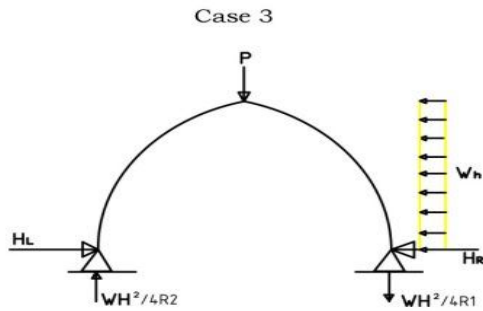


Fig 9: Horizontal Thrust for Case 3 Loading, Uniformly Distributed Horizontal Load Wh

3.2. Loading Cases

3.2.1. Case (1): Concentrated Load at Mid Span

Due to the complexity of the expression derived for the horizontal thrust it is not stated here but is given in the appendix. The value of the horizontal reaction for a concentrated load P at mid span is plotted in (Figs. 10a and 10b) in the non-dimensional form of $2H/P$ for different values of angles α and β . It can be seen that for the arches with angles of α between 5° and 20° , the horizontal thrust increases slightly with the increase in the height-to-span ratio (i.e., increase of β), whereas for values of α between 25° and 52° , the horizontal thrust decreases with the increase of this ratio.

3.2.2. Case (2): Uniformly Distributed Vertical Load Acting along Horizontal Projection

For a uniformly distributed vertical load of W acting along the horizontal projection the horizontal reaction, the expression for which is given in the appendix, is plotted in (Figs 11a and 11b) in the non-dimensional form of $H/0.75WRl$ for different values of angles α and β . It can be seen that for all values of α , the horizontal thrust decreases with the increase in the height-to-span ratio, however, this decrease becomes significantly greater as α increases.

Case 1

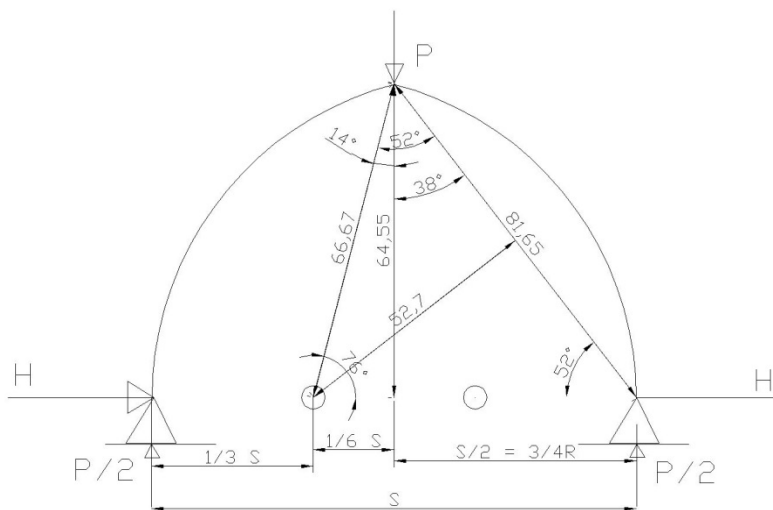
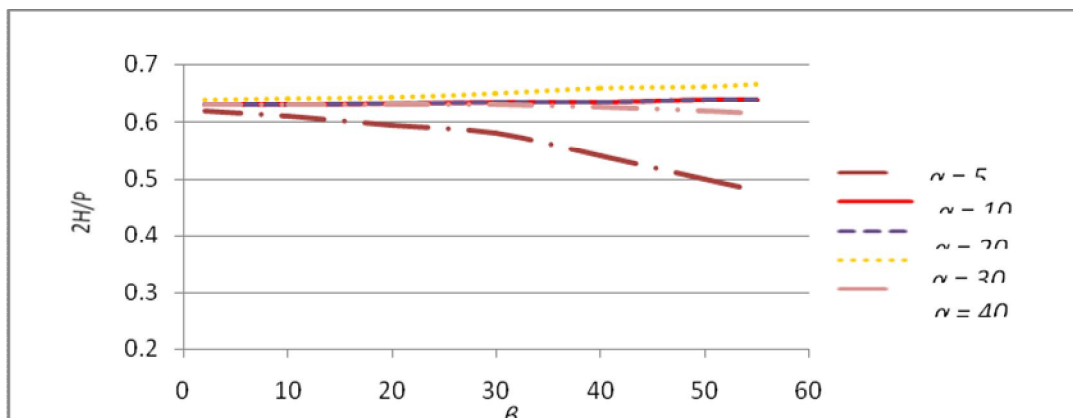


Fig 10a: Horizontal Thrust for Case 1 loading -Concentrated Load P at Mid span

Fig. 10b: Horizontal Thrust for Case 1 loading -Concentrated Load P at Mid span



Case 2

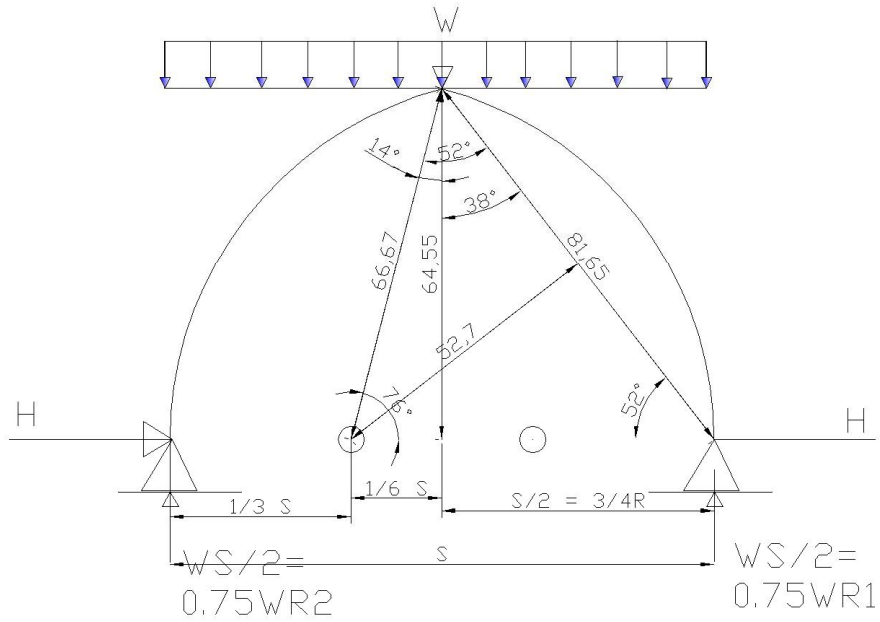


Fig 11a: Horizontal Thrust for Case 2 Loading, Uniformly Distributed Vertical Load W

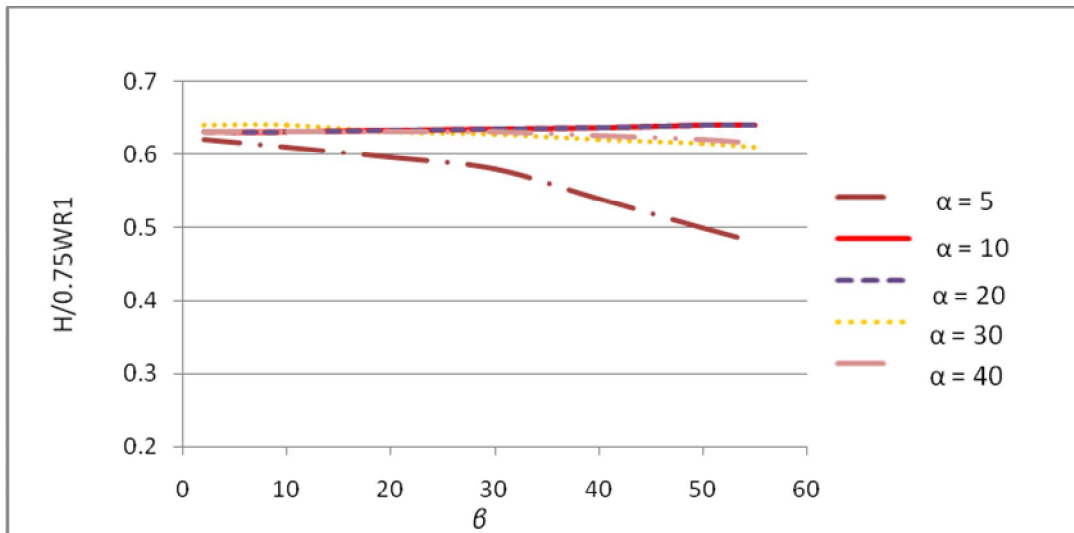


Fig 11b: Horizontal Thrust for Case 2 Loading, Uniformly Distributed Vertical Load W

3.2.3. Case (3): Uniformly Distributed Horizontal Load Acting Along Vertical Projection

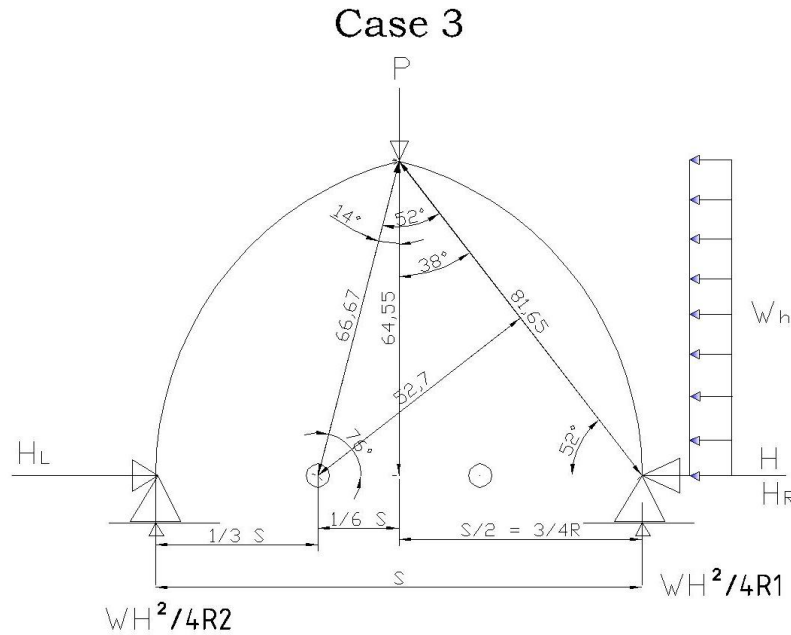


Fig 12a: Horizontal Thrust for Case 3 Loading, Uniformly Distributed Horizontal Load W_h

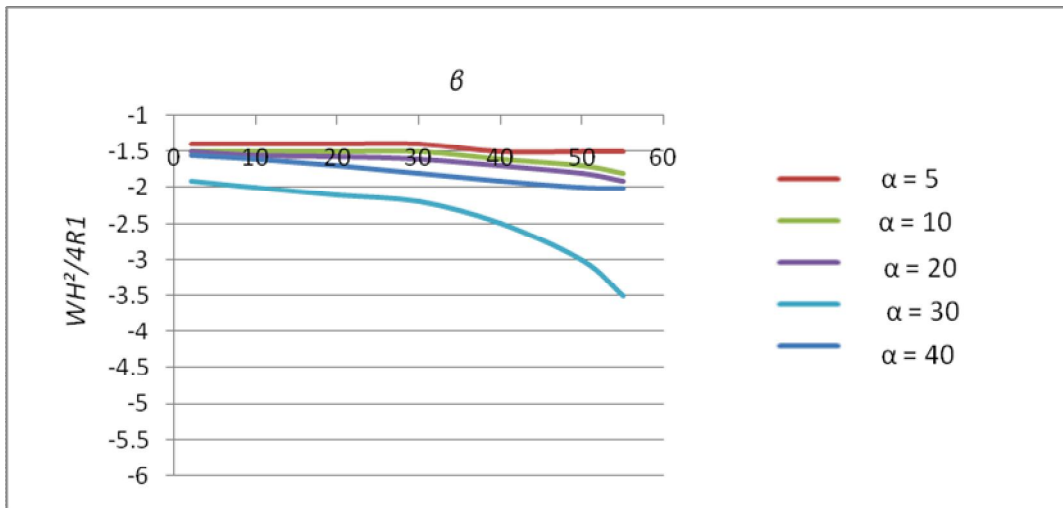


Fig 12b: Horizontal Thrust for Case 3 Loading, Uniformly Distributed Horizontal Load W_h .

For a uniformly distributed horizontal load of wh acting along the vertical projection the horizontal reaction on the side of the horizontal loading (*i.e.*, the right side), the expression for which is given in the appendix, is plotted in Figs 12a and 12b in the non-dimensional form of $wH^2/4R1$ for different values of angles α and β . Again, it can be seen that for all values of α , the horizontal thrust decreases with the increase in the height-to-span ratio, however, this decrease becomes significantly greater as α increases.

3.3 Bending Moment

Arch-type structures are most efficient if they carry their load in such a way that the funicular curve coincides with the centroid axis, which results in axial compression and no bending of the arch axis (Ziemian, 2010). Examples of arches under pure axial compression include circular arches subjected to uniform normal pressure, commonly called hydrostatic loading, parabolic arches subjected to uniform load on a horizontal projection, and centenary arches with load uniformly distributed along the arch axis.

However, for other shapes of arches and other types of loading, a small amount of bending moment will occur along the axis of the arch. Hence, arches are generally designed to resist axial compression forces and small amounts of moments.

4. In-Plane Buckling of Arches

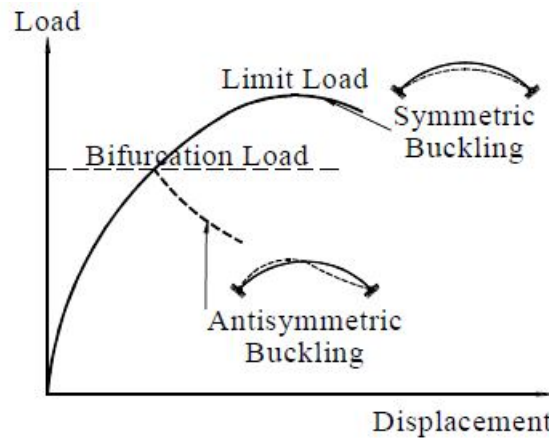


Fig 13: Modes of In-Plane arch Buckling

The stability of an arch can be characterized by buckling in or out of the plane of the arch. In plane buckling occurs when the arch is substantially braced against out-of-plane deformations, while out-of-plane buckling occurs for arches with significant free-standing portions. In-plane buckling is associated with combined compression and bending while out-of-plane buckling is associated with combined compression, biaxial bending, and torsion. This paper deals with in-plane buckling. Arches can buckle in-plane in a symmetrical buckling mode or an anti-symmetrical buckling mode, as shown in (Fig 13) (Ziemian, 2010). Generally, the symmetrical buckling load is greater than the anti-symmetrical buckling load. If an anti-symmetrical mode does not become dominant, the arch eventually becomes unstable in a symmetrical mode with the load-deflection curve gradually reaching a limit point. On the other hand, the limit load may be significantly reduced if an anti-symmetrical buckling mode dominates. This anti symmetric bifurcation load is the subject of discussion in this paper.

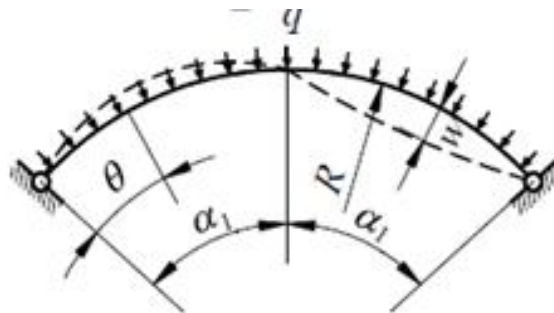


Fig 14: Typical Arch Buckling

The buckling of arches is well presented by Timoshenko and Gere (1961). The radial deflection of a circular arch of radius R and subjected to a uniform pressure is taken as w , as shown in (Fig14). The moment in the arch is assumed to be equal to the secondary moment due to the internal compressive force, S , multiplied by the deflection w (i.e., $M = Sw$). Hence, the differential equation for the buckling of the arch is

$$\frac{d^2w}{d\theta^2} + w = -\frac{R^2 S w}{EI} \quad (2)$$

Where: $S = qR$, q being the uniform pressure acting on the arch, and EI is the bending rigidity of the arch. In this equation the variation of the compressive force S along the length of the arch is neglected. Taking $k^2 = 1 + qR^3/EI$ the differential equation for the buckling of the circular arch becomes

$$\frac{d^2 w}{d\theta^2} + k^2 w = 0 \quad (3)$$

The general solution of this equation is $w = A \sin k\theta + B \cos k\theta$. Satisfying the condition at the left end ($\theta = 0$) gives $B = 0$, and the condition at the right end ($\theta = 2\alpha$) gives $\sin 2\alpha k = 0$. The smallest root for this that satisfies the condition of inextensibility of the center line of the arch is $k = \pi/\alpha$ giving

$$q_{cr} = \frac{EI}{R^3} \left(\frac{\pi^2}{\alpha_1^2} - 1 \right) \quad (4)$$

Equation (4) is also a good approximation for the case of a uniformly distributed vertical load. Austin (1971) noted that the critical thrust for the case of a two-hinged circular arch with mid span concentrated load in which large bending moment and displacements exist prior to buckling is nearly the same as the critical thrust for the uniform pressure loading which causes only compression in the arch. So, it can be assumed that the buckling data for arches subjected to loadings which cause pure compression can be used to estimate the critical loading values for other symmetrical loadings.

5- Results and Discussion

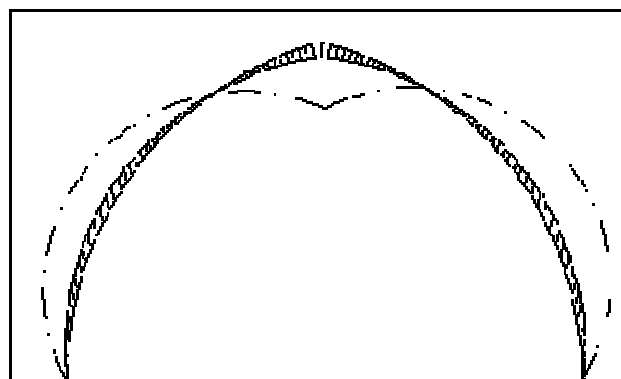
The linear analysis was used to verify the horizontal thrust reactions shown in (Figs 6 thru 9), and to obtain the bending moment diagrams. The finite element horizontal thrust compared accurately with the analytical values derived with a deviation of less than 0.5% for Case 1 loading, less than 0.3% for Case 2 loading, and less than 5% for Case 3 loading. The deformed shape and bending moments for the three cases of loading are shown in Figure 15 (a) – (c). As the pointed curve is not the funicular curve for any of these cases of loading, there is a fair amount of bending moment produced, which must be taken into consideration in the design of such arches. A geometric proportional system was used representing the arch construction phase. The proportional data process was used as a basis to calculate and to determine the precision and repeatability of the structural analysis process. The arch structure has been investigated in various studies and therefore some data exist to validate our method. Thus, our approach to reconstruct the arch allowed an accurate depiction of the structural organization and the applied procedures did not substantially change the architectural parameters. In addition, the force exerted by the arch in this study agrees well with previous force measurements indicating complete arch recruitment and likely the arch's fixation under conditions of peak force.

Furthermore, particular attention was paid to fix the arch at resting length to enable contraction at its optimal length. Also, the arch was fixed while oriented vertically, because a stretched arch would display lower tension angles and curvatures as well as reduced force exertion. Vertical deflections are commonly used as reference systems for measuring arch angles. As this study and other studies show, arches are neither planar nor static structures. Rather, lateral as well as distal bulging occurs during contraction. This presents two possible problems for the use of arches as a reference system. First, changes in shear during contraction can be the result of actual changes in the arch architecture. Second, while maintaining the forces head perpendicular to the arch surface, the arch measured before and after contraction may not be the same. This clearly hampers the comparison of relaxed and contracted arch geometry. Variations of arch angles and lengths have implications for estimates of arch characteristics. Two assumptions are made; both arch angles and lengths are homogeneous and can be represented by a single value; both parameters vary throughout arch and are hardly describable by single values. Furthermore, it should be noted that the arch is a rather simple structure and therefore estimations of the maximum force likely deviate much more in more complex arches. As this study shows, curvature represents an important parameter and should be integrated in models. Particularly close to arch attachment sites, high curvature values occur.

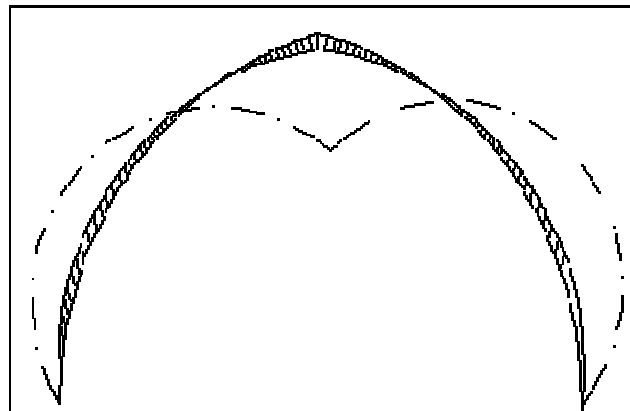
The arch curvature has further consequences when, for example, arch lengths are estimated indirectly using arch angle and distance between the fixing points because the actual length will be underestimated if the arch are curved. Besides, curvature coefficients may change not only along arch, they also vary amongst the fixing points. Hence, even though certain simplifications are necessary for practical reasons, they reflect reality better in geometrically simple arch than in complex ones. Although the approach of generating geometric arch data has been previously reported, to our knowledge this is the first time arch geometry have been used for a rigorous validation of the structural approach, and represents the first time arch property definition has been modeled in the structural properties. The approach is unique in that it represents the complex architecture of arches for specific geometric validation, and the experimental test results can be used for validation of future modeling methods as they develop. The validation approach presented based on prototype replicas focuses on the geometric issues related to arch modeling, and not the ability to incorporate the material properties into the model. The validation of models including more complex properties would be challenging, and therefore the validation approach presented here focuses on the geometric aspects of arch modeling, and can be tailored to any arch type with varying geometries. The ability of geometric models to accurately predict experimentally determined structural properties has been demonstrated previously, however those studies typically back-calculated a tissue modulus that provided the best fit between model and geometry. The principal stress on the model surface varied as a function of the particular type of surface geometry on which it was measured, and differences on the flat surface were less compared to the curved surface. The experimental surface stress data from testing the replicas can be used for assessing the validation technique which can provide important insight. In conclusion, the validation approach used here uses arch prototyping for architecture-specific validation. It will be critical to validate the apparent level and geometric mechanics which are increasingly being used, respectively, for arch prediction mechanical properties, and a basis for mechanically driven arch models of structural adaptation.

6- Conclusion

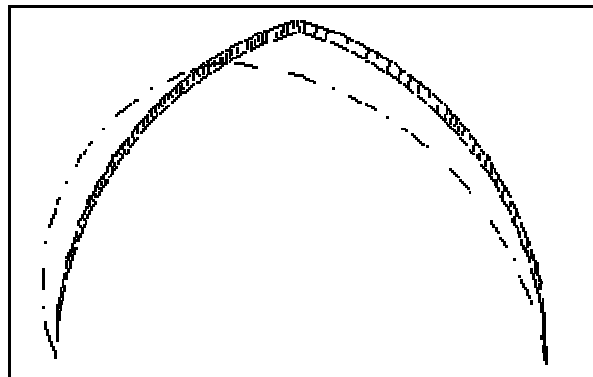
The paper presents the structural and in plane buckling analysis of two-hinged ogee arches. The geometry of ogee arches are composed of a number of interrelated variables that can be determined from the design curves presented before the analysis can be conducted. The horizontal thrust reaction of two-hinged ogee arches depends on the geometry of the arch and type of loading and, in general, decreases with the increase in height-to-span ratio. The Eigen value buckling analysis predicted an Anti-symmetric mode of buckling, which was verified by the nonlinear analysis of the arch with initial geometric imperfections. The bifurcation load of Pointed arches depend only on the height-to-base radius ratio, decreasing with the increase in the height-to-base radius ratio. The relationship between the bifurcation load and the height-to-base radius ratio is linear for concentrated mid span loads and uniform vertical loading acting along the axis of the arch. There is a fair amount of scatter in this relationship for uniform vertical loading acting along the horizontal projection, and this relationship is nonlinear for uniform horizontal loading acting along the vertical projection. The deflected shape and bending moment diagrams for the three cases of loading are shown in Fig 15. Case 1 is for a concentrated at mid-span loads; Case 2 is for uniform vertical loading; and Case 3 if a combined vertical uniform load and uniform lateral load.



Case 1: Concentrated Vertical Load



Case 2: Uniformly Distributed Vertical Load



Case 3: Uniformly distributed vertical and lateral loads

Fig 15: Deflected Shape and Bending Moment Diagram for the Three Cases of Loading

Although the limitations of this study with regards to sample size are acknowledged, the results show that architectural parameters describing the structural behavior, the deflection angles, the arch curvatures, and the arch lengths should be integrated. The incorporation and testing of these parameters in arch models will allow subsequently defining functional sub units homogeneous in their architectural properties. For example, the mid-center of the arch was architecturally more or less homogeneous and thus could be described by a fewer parameters, while the ends were more complex and would require a greater number of detail to reflect its architecture. The shear and moment aspects were more or less parallel to each other in the pointed arch both had only slightly curved shapes due to the arches tapering towards the attachment sites. In lateral direction, the arches were also curved due to the cylindrical shape of the arch. However, both shear and moment exhibited a planar central area. Due to contraction, is a very symmetrical arch, its bulging was symmetrical along they- and z-axes because the bulging affected mainly the arch's center. Thus, the curvature of arches increased in all three planes, displaying their greatest deformation at the arch's center. Compression, torsion and bending tests performed to determine the arch properties, in addition to compression testing for apparent and local level stress, demonstrated a linear region for the arch model. The strain rate tests confirmed a linear elastic stress-strain relation. On the other hand, the analytical calculations serving as a benchmark are also limited due to the use of modulus and the calculations are sensitive to their representation of a round cylindrical surface. Nevertheless, taking into consideration the limitations of each measurement technique, the comparison of those model results with the experimental results are presented.

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